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Graphs with most number of three point induced connected subgraphs

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Abstract

Let G be a simple graph with e perfectly reliable edges and n nodes which fail independently and with the same probability ρ. The residual connectedness reliability $R(G,\rho)$ of G is the probability that the graph induced by the surviving nodes is connected. If $\Gamma(n,e)$ is the collection of all n node e edge simple graphs, then $G \in \Gamma(n,e)$ is uniformly most reliable if $R(G,\rho) \geqslant R(G',\rho)$ for all $G' \in \Gamma(n,e)$ and all $0 < \rho < 1$. If $S_3(G)$ is the number of three point induced connected subgraphs of G, then $G \in \Gamma(n,e)$ is S_3-maximum if $S_3(G) \geqslant S_3(G')$ for all $G' \in \Gamma(n,e)$. It is known that if $G \in \Gamma(n,e)$ is S_3-maximum and ρ is sufficiently large then $R(G,\rho) > R(G,\rho')$ for all non-S_3-maximum graphs $G' \in \Gamma(n,e)$. This paper characterizes the S_3-maximum graphs in $\Gamma(n,e)$ for the range $e \leqslant (n^2/4) + (2n-3)/4$.

1. Introduction

Consider the following network reliability measure defined on simple graphs in which the edges are perfectly reliable and the points fail independently of each other. Let $p(u)$ be the probability that point u operates; equivalently $\rho(u) = 1 - p(u)$ is the failure probability of u. We say that a given network is in an operational state if the surviving points induce a connected subgraph. The residual connectedness reliability of a network G, denoted by $R(G,\rho)$, is the probability that the graph induced by the surviving points is connected.

For purpose of contrast, we describe another well-known and important node failure model, called the K-terminal connectedness model. In this model, all edges of a simple graph G and all points from a specified subset K of the vertices of G are perfectly reliable; the remaining nodes operate independently of one another with probabilities denoted as above. The network is said to be in an operational state if the surviving nodes induce a subgraph of G in which all the nodes of K lie in a common

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connected component. The residual connectedness model is different from the K-terminal connectedness model in the sense that it is not a hierarchical system. Specifically, let E be a finite set and $\wp(E)$ be the set of subsets of E. A system (E, Ω) consists of E and a collection of operating states $\Omega \subseteq \wp(E)$. A hierarchical system (E, Ω) is one where Ω is closed upward with respect to set inclusion, i.e., a superset of an operating state is an operating state. We say that the system is operational if the collection of operating components is an operating state of the system. Assuming a probability distribution $\mathbf{Pr}$ on $\wp(E)$, the reliability of the system is just $\mathbf{Pr}(\Omega)$. It is easily seen that the K-terminal connectedness model is hierarchical. The residual connectedness is not hierarchical since a supergraph of a connected graph may be disconnected.

An important special case arises when the failure probabilities of the nodes are all equal to ρ, $0 < \rho < 1$. Let $\Gamma(n,e)$ be the collection of graphs on n points and e edges. We say that a graph $G \in \Gamma(n,e)$ is uniformly most reliable if $R(G,\rho) \geqslant R(G',\rho)$ for all $G' \in \Gamma(n,e)$ and all $0 < \rho < 1$. One of the important open problems in residual connectedness reliability is the characterization of uniformly most reliable graphs. Unfortunately, such graphs do not exist for all possible values of n and e. To see this, consider the reliability polynomial $R(G,\rho)$ of the residual connectedness reliability of $G \in \Gamma(n,e)$,

$$R(G,\rho) = \sum_{i=1}^{n} S_i(G)\rho^{n-i}(1-\rho)^i, \tag{1}$$

where $S_i(G)$ is the number of connected point induced subgraphs of G with i points. The coefficients S_1 and S_2 in the expression (1) are clearly n and e respectively. Moreover, if $\kappa(G)$ is the minimum number of points whose deletion disconnects G, then for $i > n - \kappa(G)$ we have $S_i(G) = \binom{n}{i}$. Thus (1) can be written as

$$\begin{aligned} R(G,\rho) &= n(1-\rho)\rho^{n-1} + e(1-\rho)^2\rho^{n-2} + E(G,\rho) \\ &\quad + \sum_{i=n-\kappa(G)+1}^{n} \binom{n}{i} \rho^{n-i}(1-\rho)^i, \end{aligned} \tag{2}$$

where

$$E(G,\rho) = \sum_{i=3}^{n-\kappa(G)} S_i(G)\rho^{n-1}(1-\rho)^i.$$

Clearly the terms in (2) except $E(G,\rho)$ are the same for any two graphs $G_1, G_2 \in \Gamma(n,e)$ when $\kappa(G_1) = \kappa(G_2)$. Frank [2,3] considered the problem of characterizing the most reliable graphs for sufficiently small ρ. A graph $G \in \Gamma(n,e)$ is κ-maximum if $\kappa(G) \geqslant \kappa(G')$ for all $G' \in \Gamma(n,e)$. A κ-maximum graph $G \in \Gamma(n,e)$ is κ-optimal if $S_{n-k}(G) \geqslant S_{n-k}(G')$ for all κ-maximum graphs $G' \in \Gamma(n,e)$. Frank showed that if $G \in \Gamma(n,e)$ is κ-optimal and ρ is sufficiently small then $R(G,\rho) \geqslant R(G',\rho)$ for all non-κ-optimal graphs $G' \in \Gamma(n,e)$. Thus one of the necessary conditions for uniformly most reliable graphs is κ-optimality. We say that a graph $G \in \Gamma(n,e)$ is S_3-maximum if $S_3(G) \geqslant S_3(G')$ for all $G' \in \Gamma(n,e)$. Using Frank's approach it can readily be seen that if $G \in \Gamma(n,e)$ is S_3-maximum and ρ is sufficiently large then $R(G,\rho) \geqslant R(G',\rho)$ for all non-S_3-maximum graphs $G' \in \Gamma(n,e)$. Thus, a second necessary condition is obtained, namely uniformly most reliable graphs are S_3-maximum. Now consider the class $\Gamma(n,e)$ such that $e = n$. It can be easily shown that the cycle C_n on n points is the only κ-optimal graph in $\Gamma(n,n)$, but C_n is not S_3-maximum. Therefore no uniformly best graph exists in $\Gamma(n,n)$. However, there are values of n and e such that $G \in \Gamma(n,e)$ is both κ-optimal and S_3-maximum. For example, the complete bipartite graph $K_{p,p}$ is both κ-optimal and S_3-maximum in $\Gamma(2p,p^2)$. Hence, any characterization of uniformly best reliable graphs must first address the issue of characterizing the graphs that are both κ-optimal and S_3-maximum. Moreover, S_3-maximum graphs are of considerable importance in themselves since the most reliable graphs for sufficiently large ρ are S_3-maximum graphs. The S_3-maximum graphs in $\Gamma(n,e)$ for the cases $0 \leqslant e \leqslant 2(n-2)$ and $2(n-2) \leqslant e \leqslant 3(n-3)$ have been characterized by Boesch and Li [1] and Salizkiy [5]. We characterize S_3-optimal graphs in the range $e \leqslant (n^2/4) + (2n-3)/4$.

2. Preliminaries

Unless defined otherwise, the graph theoretic terminology used here follows Harary [4]. If n is a positive integer, we will denote by $[n]$ the set $\{1,2,\dots,n\}$. Let $G=(V,E)$ be a graph with $V=[n]$. By $e(G)$ we mean $|E(G)|$. If $A \subseteq [n]$ then the subgraph induced by A in G is denoted by $\langle A \rangle_G$. Let $\Gamma(n)$ be the collection of all graphs with $V(G)=[n]$. Likewise, $\Gamma(n,e)$ is the set of all graphs G with $V(G)=[n]$ and $|E(G)|=e$. Let $\wp([n])$ be the set of subsets of $[n]$ and $\wp_3([n])$ the set of all 3-element subsets of $[n]$. We denote by $\Gamma_3(n)$ the collection of all subsets of $\wp_3([n])$. Each object in $\Gamma_3(n)$ can be regarded as a hypergraph on the vertex set $[n]$ whose edges all have cardinality 3. A subset A of $[n]$ is an *odd triple* of G if A has three elements and the subgraph $\langle A \rangle_G$ induced by A on G has an odd number of edges. The collection of all odd triples of G is denoted by $T(G)$. The number of odd triples $|T(G)|$ is denoted by $\tau(G)$. The number of odd triples incident on a vertex v of G will be denoted by $\tau_v(G)$ and $\tau_{\min}(G) = \min\{\tau_v(G)\colon v \in [n]\}$. Let $t_j(G)$, where $0 \leqslant j \leqslant 3$, be the number of 3-subsets of $[n]$ that induce a subgraph in G having exactly j edges. Let $S_3(G)$ be the number of 3-subsets of $[n]$ that induce a connected subgraph in G. A *two-graph* on a set of points X is a collection U of 3-subsets of X such that every 4-subset of X contains an even number of members of U. Two-graphs were proposed in 1970 by Graham Higman, and studied by D.E. Taylor. For a survey of results on two-graphs see [7,8]. The collection of odd triples of a graph is a two-graph. Conversely, every two-graph arises as the collection of odd triples of some graph. A graph $G \in \Gamma(n,e)$ is called S_3-*maximum* if $S_3(G) \geqslant S_3(G')$ for all $G' \in \Gamma(n,e)$. A graph $G \in \Gamma(n,e)$ is called τ-*minimum* if $\tau(G) \leqslant \tau(G')$ for all $G' \in \Gamma(n,e)$.

Proposition 2.1. *If G is a graph on n vertices and e edges then* $S_3(G) = (e(n-2) - \tau(G))/2$.

Proof. Let $\sigma(G) = \{(x, H)$: H is a three-point induced subgraph of G and $x \in E(H)\}$. As each edge of G belongs to exactly $n-2$ subgraphs induced by three points of G, we have $|\sigma(G)| = e(n-2)$. However, $|\sigma(G)| = t_1(G) + 2t_2(G) + 3t_3(G)$. Hence $e(n-2) = t_1(G) + t_3(G) + 2(t_2(G) + t_3(G)) = \tau(G) + 2S_3(G)$. □

An important consequence of Proposition 2.1 is the fact that a graph $G \in \Gamma(n, e)$ is S_3-maximum iff G is τ-minimum.

Corollary 2.1. *A graph G on n vertices and $e = j(n-j)$ edges is S_3-maximum iff G is the complete bipartite graph $K_{j,n-j}$.*

Proof. Clearly, a graph G is a complete bipartite graph iff $\tau(G) = 0$. The corollary follows from Proposition 2.1. □

A graph G is said to be *almost regular* if $|\deg(u) - \deg(v)| \leqslant 1$ for all vertices u, v of G.

Corollary 2.2. *If G is an almost regular complete multipartite graph on n points and e edges then G is the unique S_3-maximum graph among all graphs on n points and e edges.*

Proof. By Proposition 2.1 and the fact that

$$S_3(G) = \sum_{v \in V(G)} \binom{\deg(v)}{2} - 2t_3(G),$$

it follows that $S_3(G) = (2e(2n-3) - \sum_{v \in V(G)} (\deg(v))^2 - 4t_1(G))/6$. If G is an almost regular complete multipartite graph then $t_1(G) = 0$. Moreover, $\sum_{v \in V(G)} (\deg(v))^2$ is minimum among all n point e edge graphs. Conversely, if these two conditions are satisfied by a graph, then it is an almost regular complete multipartite graph. Indeed, $t_1(G) = 0$ is equivalent to $t_2(\bar{G}) = 0$, where $\bar{G}$ is the complement of G. Also, $\sum_{v \in V(G)} (\deg_{\bar{G}}(v))^2$ is minimum. If $t_2(\bar{G}) = 0$ then $\bar{G}$ is a disjoint union of cliques. From the minimality of the sum of the squares of the degrees we conclude $\bar{G}$ is almost regular. Hence G, being the complement of an almost regular graph which is a disjoint union of cliques, must be an almost regular complete multipartite graph. □

Two edges of a graph are said to be *independent* if they do not have any endpoints in common. Let $i_2(G)$ denote the number of unordered pairs of independent edges of G. The following corollary is a consequence of Proposition 2.1 and the fact that

$$S_3(G) = \binom{e}{2} - i_2(G) - 2t_3(G)).$$

Corollary 2.3. *If $G \in \Gamma(n,e)$, then $\tau(G) = e(n-1-e) + 2i_2(G) + 4t_3(G)$.*

An immediate consequence of Corollary 2.3 is the following corollary.

Corollary 2.4. *If $G \in \Gamma(n,e)$, then $\tau(G) \geqslant e(n-1-e)$. Moreover $\tau(G) = e(n-1-e)$ iff G is a spanning subgraph of $K_{1,n-1}$.*

3. S_3-maximum graph

If X and Y are sets then by $X \oplus Y$ we mean the symmetric difference of X and Y, namely $(X \cup Y) - (X \cap Y)$. If $G, G' \in \Gamma(n)$ then by $G \oplus G'$ we mean the graph with $V(G \oplus G') = [n]$ and $E(G \oplus G') = E(G) \oplus E(G')$. By $G \cap G'$ we mean the graph with $V(G \cap G') = [n]$ and $E(G \cap G') = E(G) \cap E(G')$. Note that $\Gamma(n)$ and $\Gamma_3(n)$ are commutative groups with respect to the operation $\oplus$.

Recall that $T(G)$ is the collection of all triples of G. We have the following lemma.

Lemma 3.1. *If $G, G' \in \Gamma(n)$, then $T(G \oplus G') = T(G) \oplus T(G')$.*

Proof. Let A be a 3-subset of $[n]$. Then $e(\langle A \rangle_{G \oplus G'}) = e(\langle A \rangle_G) + e(\langle A \rangle_{G'}) - 2e(\langle A \rangle_{G \cap G'})$. Therefore $e(\langle A \rangle_{G \oplus G'})$ is odd iff exactly one of the numbers $e(\langle A \rangle_G)$, $e(\langle A \rangle_{G'})$ is odd. Thus $A \in T(G \oplus G')$ iff $A \in T(G) \oplus T(G')$. $\square$

If $A \subseteq [n]$ then by $B(A)$ we mean the complete bipartite graph with bipartitions A and $[n] - A$. Note that $B(\emptyset) = B([n])$ is the edgeless graph on n points. The collection of all complete bipartite graphs on vertex set $[n]$ is denoted by $\beta(n)$. Note that $\beta(n)$ with the operation $\oplus$ is a subgroup of $\Gamma(n)$. If $G \in \Gamma(n)$ and $v \in [n]$, the neighborhood of v in G will be denoted by $N_G(v)$. The proof of Corollary 2.1 can be rephrased as follows.

Lemma 3.2. *$T(G) = \emptyset$ iff $G \in \beta(n)$.*

Corollary 3.1. *$\beta(n)$ is closed under $\oplus$.*

Proof. If G, $G' \in \beta(n)$ then by Lemma 3.1 $T(G \oplus G') = T(G) \oplus T(G') = \emptyset \oplus \emptyset = \emptyset$. Therefore, by Lemma 3.2, $G \oplus G' \in \beta(n)$. $\square$

In $\Gamma(n)$ we define the relation $\equiv$ by $G \equiv G'$ iff $G \oplus G' \in \beta(n)$ and we say that G and G' are *switching equivalent*. Since $\beta(n)$ is closed under $\oplus$, and $G \oplus G = \emptyset$ for all $G \in \Gamma(n)$, it follows that $\equiv$ is an equivalence relation. The equivalence classes under this relation are just the "switching classes" introduced by Van Lint and Seidel [9] in their study of equilateral point sets in elliptic geometry. Seidel [6] also used switching classes in the study of strongly regular graphs. The following corollary is immediate from Lemmas 3.1 and 3.2.

Corollary 3.2. *If $G \equiv G'$ then $T(G) = T(G')$.*

The following lemma shows how to choose a representative from each equivalence class in a canonical way with respect to a fixed vertex v.

Lemma 3.3. *If $G \in \Gamma(n)$ and $v \in [n]$, then there is a unique $G' \equiv G$ such that v is an isolated point of G'.*

Proof. Define $G' = G \oplus B(N_G(v))$. Clearly $G' \equiv G$ and v is an isolated point of G'. Moreover, if $G'' \equiv G$ and v is an isolated point of G'', then $G' \oplus G'' \in \beta(n)$ and v is an isolated point of $G' \oplus G''$. Thus $G' \oplus G'' = \emptyset$, and $G' = G''$. □

The unique $G' \equiv G$ such that v is an isolated point of G' will be denoted by $\pi_v(G)$. Since $T(G) = T(\pi_v(G))$, it follows that $\tau_v(G) = \tau_v(\pi_v(G)) = e(\pi_v(G))$. If e is a non-negative integer, define

$$\|e\| = \min\{|e - i(n-i)|: 0 \leqslant i \leqslant \lfloor n/2 \rfloor\}.$$

Lemma 3.4. *If $G \in \Gamma(n,e)$ then $\|e\| \leqslant \tau_{\min}(G)$.*

Proof. Since $G \equiv \pi_v(G)$, there exists some $B \in \beta(n)$ such that $G = \pi_v(G) \oplus B$. Hence $e(G) = e(B) + e(\pi_v(G)) - 2e(B \cap \pi_v(G))$. Thus $\|e\| \leqslant |e(G) - e(B)| \leqslant |e(\pi_v(G)) - 2e(B \cap \pi_v(G))| \leqslant e(\pi_v(G)) = \tau_v(G)$. The first inequality follows from the definition of $\|e\|$, while the last inequality is a consequence of the fact that $e(B \cap \pi_v(G)) \leqslant e(\pi_v(G))$. Since the vertex v is arbitrary, the lemma follows. □

Lemma 3.5. *If $G \in \Gamma(n)$ then $\tau(G) \geqslant \tau_v(G)(n - 1 - \tau_v(G))$ for any vertex v of G.*

Proof. The lemma follows immediately from Corollary 2.4 after observing that $\tau(G) = \tau(\pi_v(G))$ and $\pi_v(G)$ has $\tau_v(G)$ edges. □

Theorem 3.1. *If $G \in \Gamma(n,e)$ then $\tau(G) \geqslant \|e\|(n-1-\|e\|)$. Moreover, $\tau(G) = \|e\|(n-1-\|e\|)$ iff $G \equiv S$,* where S is a spanning subgraph of $K_{1,n-1}$ *having $\|e\|$ edges.*

We need the following lemmas in the proof of Theorem 3.1.

Lemma 3.6. *If $G \in \Gamma(n,e)$ then $\tau(G) \geqslant e(n - 2\Delta(G))$ where $\Delta(G)$ is the maximum degree of G.*

Proof. By Proposition 2.1 and the fact that

$$S_3(G) = \sum_{v \in V(G)} \binom{\deg(v)}{2} - 2t_3(G),$$

it follows that $\tau(G) = en + 4t_3(G) - \sum_{v\in V(G)}(\deg(v))^2$. Hence $\tau(G) \geqslant en - \sum_{v\in V(G)}(\deg(v))^2 \geqslant en - \sum_{v\in V(G)}(\Delta(G)\deg(v)) = e(n - 2\Delta(G))$. $\square$

Lemma 3.7. *If $G \in \Gamma(n,e)$ then $\tau_v(G) \geqslant \deg(v)(n - \deg(v)) - e$.*

Proof. Let $B = B(N_G(v))$. Then $\tau_v(G) = \tau_v(G \oplus B) = |E \oplus B| = e + |E(B)| - 2|E(G \cap B)| \geqslant |E(B)| - e = \deg(v)(n - \deg(v)) - e$. $\square$

Lemma 3.8. *If $G \in \Gamma(n)$ and $\tau_{\min}(G) > (n-1)/2$ then $\tau(G) > (n-1)^2/4$.*

Proof. If $\tau_{\min}(G) \geqslant 3(n-1)/4$ then $\tau(G) \geqslant (n_{\min}(G))/3 > (n-1)^2/4$. Hence, assume that $(n-1)^2/2 < \tau_{\min}(G) < 3(n-1)/4$. Let v be a point of G such that $\tau_v(G) = \tau_{\min}(G)$. By Lemma 3.3, there exists a unique $G' \in \Gamma(n)$ such that $G' \equiv G$ and v is an isolated point of G' Moreover, by Corollary 3.2, it follows that $\tau_{\min}(G) = \tau_{\min}(G')$ and $\tau(G) = \tau(G')$. Without loss of generality, we may assume that v is an isolated point of G, for otherwise we can replace G by G'. As v is an isolated point of G, we conclude that $\tau_{\min}(G) = |E(G)|$. We have two cases.

Case 1: $\Delta(G) \leqslant \tau_{\min}(G) - (n-2)/2$. Since $\tau_{\min}(G) < 3(n-1)/4$, it follows that $\Delta(G) < (n+1)/4$. By Lemma 3.6, $\tau(G) \geqslant \tau_{\min}(G)(n - 2\Delta(G)) > (n-1)^2/4$.

Case 2: $\Delta(G) > \tau_{\min}(G) - (n-2)/2$. As $\Delta(G)$ is an integer, this case is equivalent to $\Delta(G) \geqslant \tau_{\min}(G) - (n-3)/2$.

We first claim that $\Delta(G) \leqslant (n-1)/2$. Suppose otherwise. Let u be a point of G with $\deg(u) = \Delta(G)$. If each $w \in N_G(u)$ has $\deg(w) \geqslant 2$, then $\tau_{\min}(G) \geqslant (\Delta(G) + 2\Delta(G))/2 = 3\Delta(G)/2 > 3(n-1)/4$. This is a contradiction to the hypothesis that $\tau_{\min}(G) < 3(n-1)/4$. Hence, there is some $w \in N_G(u)$ with $\deg(w) = 1$. Consider the graph $G' = G \oplus B(\{u\})$. Clearly w is an isolated point of G' and $G' \equiv G$. Thus, by Corollary 3.2, $\tau_w(G) = \tau_w(G') = |E(G')|$. But $|E(G')| = |E(G-u)| + n - 1 - \Delta(G)$. Since $\Delta(G) > (n-1)/2$, $\tau_w(G) < |E(G-u)| + (n-1)/2 < |E(G-u)| + \Delta(G) = \tau_{\min}(G)$. Thus $\tau_w(G) < \tau_{\min}(G)$, a contradiction. Hence $\Delta(G) \leqslant (n-1)/2$.

We continue denoting by v an isolated point of G with $\tau_{\min}(G) = \tau_v(G)$ and also continue denoting by u a point of G with $\Delta(G) = \deg(u)$. Clearly $v \neq u$. Then $\tau(G) = \tau_u(G) + \tau(G-u)$ and by Lemma 3.7, $\tau(G) \geqslant \Delta(G)(n - \Delta(G)) - |E(G)| + \tau(G-u) = \Delta(G)(n - \Delta(G)) - \tau_{\min}(G) + \tau(G-u)$. By Corollary 2.4, $\tau(G-u) \geqslant |E(G-u)|(n - 2 - |E(G-u)|)$. Thus

$$\begin{aligned}\tau(G) &\geqslant \Delta(G)(n - \Delta(G)) - \tau_{\min}(G) + (\tau_{\min}(G) - \Delta(G))(n - 2 - \tau_{\min}(G) + \Delta(G))\\ &= \tau_{\min}(G)(n - 1 - \tau_{\min}(G)) + 2(\Delta(G) - 1)(\tau_{\min}(G) - \Delta(G)).\end{aligned}$$

The function $f(x) = (x-1)(\tau_{\min}(G) - x)$, in the interval $\tau_{\min}(G) - (n-3)/2 \leqslant x \leqslant (n-1)/2$, is minimum at $x = (n-1)/2$. Hence, $\tau(G) \geqslant \tau_{\min}(G)(n - 1 - \tau_{\min}(G)) + (n-3)(\tau_{\min}(G) - (n-1)/2)$. The function $g(x) = x(n-1-x) + (n-3)(x - (n-1)/2)$ is strictly increasing for $x \leqslant n-2$. Since $\tau_{\min}(G)$ is an integer, we may write the hypothesis $(n-1)/2 < \tau_{\min}(G) < 3(n-1)/4$ as $(n-1)/2 < \tau_{\min}(G) \leqslant (3n-4)/4$.

As $(3n-4)/4 \leqslant n-2$ for $n \geqslant 4$, it follows that $\tau(G) \geqslant g(\tau_{\min}(G)) > g((n-1)/2) = (n-1)^2/4$. $\square$

Proof of Theorem 3.1. We distinguish two cases.

Case 1: $\tau_{\min}(G) \leqslant (n-1)/2$.

Since the function $f(x) = x(n-1-x)$ is strictly increasing for $x \leqslant (n-1)/2$, it follows, by Lemmas 3.4 and 3.5, that $\tau(G) \geqslant \tau_{\min}(G)(n-1-\tau_{\min}(G)) \geqslant \|e\|\ (n-1-\|e\|)$.

Case 2: $\tau_{\min}(G) > (n-1)/2$. Then, by Lemma 3.8, we have $\tau(G) > (n-1)^2/4 \geqslant \|e\|\ (n-1-\|e\|)$.

This establishes the first part of the theorem. In addition, if $\tau(G) = \|e\|\ (n-1-\|e\|)$, it must be the case that $\tau_{\min}(G) \leqslant (n-1)/2$, and since $\tau(G) \geqslant \tau_{\min}(G)\ (n-1-\tau_{\min}(G)) \geqslant \|e\|(n-1-\|e\|)$, we must have $\tau_{\min}(G) = \|e\|$. Let v be a point of G with $\tau_v(G) = \tau_{\min}(G)$. By Lemma 3.3, there exists a graph G' such that $G' \equiv G$ and v is isolated in G'. By Corollary 3.2 it follows that $\tau(G) = \tau(G')$ and $\tau_v(G) = \tau_v(G') = |E(G')|$. Thus $\tau(G') = \|e\|(n-1-\|e\|)$ and $|E(G')| = \|e\|$. By Corollary 2.4, G' must then be a spanning subgraph of $K_{1,n-1}$. This proves the second part of the theorem. $\square$

If $G \in \Gamma(n,e)$, the lower bound on $\tau(G)$ given by Theorem 3.1 cannot be attained unless $e \leqslant n^2/4 + (2n-3)/4$. Indeed, according to Theorem 3.1, a graph $G \in \Gamma(n,e)$ with $\tau(G) = \|e\|(n-1-\|e\|)$ must be switching equivalent to a spanning subgraph S of $K_{1,n-1}$, i.e., there must be some complete bipartite graph B such that $G = B \oplus S$. Suppose B has bipartitions W_1, W_2 with $|W_1| \leqslant |W_2|$. Then $e \leqslant |W_2|\ (n-|W_2|) + |W_2| - 1$. The function $f(x) = x(n-x) + x - 1$ is maximum at $x = (n+1)/2$. Thus $e \leqslant f(|W_2|) \leqslant f((n+1)/2) = n^2/4 + (2n-3)/4$. Our next theorem shows that the lower bound on $\tau(G)$ given by Theorem 3.1 can always be attained whenever $e \leqslant n^2/4 + (2n-3)/4$.

Theorem 3.2. *If $e \leqslant n^2/4 + (2n-3)/4$ then there exists a graph $G \in \Gamma(n,e)$ such that $\tau(G) = \|e\|(n-1-\|e\|)$.*

Proof. We have three cases.

Case 1: $e = j(n-j)$ for some j such that $0 \leqslant j \leqslant \lfloor n/2 \rfloor$. Let B the bipartite graph with bipartitions of sizes j and $n-j$. Then $B \in \Gamma(n,e))$, $\|e\| = 0$, and $\tau(B) = 0$ by Corollary 2.1.

Case 2: $j(n-j) < e < (j+1)(n-j-1)$ for some j, $0 \leqslant j < \lfloor n/2 \rfloor$. Then $\|e\| \leqslant ((j+1)(n-j-1) - j(n-j))/2 \leqslant n-j-1$ for $n \geqslant 1$. We have $\|e\| = e - j(n-j)$ or $\|e\| = (j+1)(n-j-1) - e$.

Case 2.1: $\|e\| = e - j(n-j)$. Let B be the bipartite graph with partitions $\{1,2,\dots,j\}$ and $\{j+1,\dots,n\}$ and let $S = (V,E)$ with $V = \{1,2,\dots,n\}$ and $E = \{\{j+1, j+k\}: 2 \leqslant k \leqslant \|e\| + 1\}$. Set E is well-defined since $\|e\| + 1 \leqslant n-j$.

S is a spanning subgraph of $K_{1,n-1}$ having $\|e\|$ edges. Let $G = B \oplus S$. Then $G \in \Gamma(n, e)$ and $\tau(G) = \tau(S) = \|e\|(n - 1 - \|e\|)$.

Case 2.2: $\|e\| = (j + 1)(n - j - 1) - e$. Let B be the bipartite graph with bipartitions $\{1, 2, \ldots, j + 1\}$ and $\{j + 2, \ldots, n\}$. Let $S = (V, E)$ with $V = \{1, 2, \ldots, n\}$ and $E = \{\{1, j + k\}: 2 \leqslant k \leqslant \|e\| + 1\}$. Again E is well-defined because $\|e\| + 1 \leqslant n - j$. S is a spanning subgraph of $K_{1,n-1}$ with $\|e\|$ edges. Let $G = B \oplus S$. We have $G \in \Gamma(n, e)$ and $\tau(G) = \tau(S) = \|e\|(n - 1 - \|e\|)$.

Case 3: $e > \lfloor n/2 \rfloor \lceil n/2 \rceil$. Then $\|e\| = e - \lfloor n/2 \rfloor \lceil n/2 \rceil$. It can be easily shown that $\lfloor n/2 \rfloor \lceil n/2 \rceil < e \leqslant n^2/4 + (2n - 3)/4$ implies $\|e\| \leqslant \lceil n/2 \rceil - 1$. Let B be the complete bipartite graph with bipartitions $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$ and $\{\lfloor n/2 \rfloor + 1, \ldots, n\}$. Let $S = (V, E)$ with $V = \{1, 2, \ldots, n\}$ and $E = \{\{\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + k\}: 2 \leqslant k \leqslant \|e\| + 1\}$. Since $\|e\| \leqslant \lceil n/2 \rceil - 1$, it follows that $\lfloor n/2 \rfloor + \|e\| + 1 \leqslant \lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. Hence E is well-defined. Let $G = B \oplus S$. Then $G \in \Gamma(n, e)$ and $\tau(G) = \tau(S) = \|e\|(n - 1 - \|e\|)$. $\square$

The next theorem shows that the graphs constructed in the proof of Theorem 3.2 are essentially the only τ-minimum graphs in the range under consideration.

Theorem 3.3. *If $G \in \Gamma(n, e)$ and $e \leqslant n^2/4 + (2n - 3)/4$ then G is τ-minimum iff $G = B \oplus S$ and $B \cap S$ is either $\emptyset$ or S, where S is a spanning subgraph of $K_{1,n-1}$ having $\|e\|$ edges and B is a complete bipartite graph such that $\|e\| = |e(G) - e(B)|$.*

Proof. Suppose that $G \in \Gamma(n, e)$ and $G = B \oplus S$ where S is a spanning subgraph of $K_{1,n-1}$ having $\|e\|$ edges and B is a complete bipartite graph on vertex set $\{1, 2, \ldots, n\}$. Then $\tau(G) = \tau(S) = \|e\|(n - 1 - \|e\|)$. Hence, by Theorem 3.1, G is τ-minimum. Conversely, suppose $G \in \Gamma(n, e)$, with $e \leqslant n^2/4 + (2n - 3)/4$, is τ-minimum. By Theorems 3.1 and 3.2, it follows that $\tau(G) = \|e\|(n - 1 - \|e\|)$. Hence, by Theorem 3.1, $G \equiv S$ where S is a spanning subgraph of $K_{1,n-1}$ having $\|e\|$ edges. Thus, $G = B \oplus S$, where B is a complete bipartite graph on vertex set $[n]$. Since $e(G) = e(B) + e(S) - 2e(B \cap S)$, we have $\|e\| \leqslant |e(G) - e(B)| = |e(S) - 2e(B \cap S)| \leqslant e(S) = \|e\|$. Therefore $|e(G) - e(B)| = \|e\|$ and $|e(S) - 2e(B \cap S)| = e(S)$. This implies $B \cap S = \emptyset$ or $B \cap S = S$. $\square$

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